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ARONSZAJN NULL AND GAUSSIAN NULL SETS COINCIDE

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ABSTRACT

We prove that in every separable Banach space the σ ideals of Aronszajn null sets, Gaussian null sets and cube null sets coincide.

In order to prove Gâteaux differentiability of Lipschitz mappings between Banach spaces, various authors introduced, sometimes implicitly, different notions of 'null sets' in separable Banach spaces. In all these cases, the null sets form a proper σ ideal of subsets of the given separable Banach space B and the differentiability result says that Lipschitz mappings from B to Banach spaces having the Radon–Nikodym property are Gâteaux differentiable almost everywhere with respect to it.

The first such results were due to Christensen [4] and to Mankiewicz [5]. By showing that the differentiability result holds almost everywhere with respect to non-degenerate cube measures, Mankiewicz implicitly introduced the σ ideal of **cube null sets** in B as the family of those Borel subsets of B that are null for every non-degenerate cube measure. (Non-degenerate cube measures in B may be defined as distributions of the random variables of the form $a + \sum_k X_k e_k$, where $a, e_1, e_2, \ldots \in B$, $\sum_k \|e_k\| < \infty$, the span of $e_1, e_2, \ldots$ is dense in B, and X_k are uniformly distributed independent random variables with values in $[0, 1]$.) Christensen's approach is based on the observation that the σ ideal of **Haar null sets** in an abelian locally compact polish group B may be defined without referring to the Haar measure by: A Borel set E in B is Haar null if

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there is a Borel probability measure μ on B such that every translate of E has μ measure zero. This statement may be therefore used as a definition of the notion of Haar null sets in Banach spaces (or, more generally, in polish abelian groups). The σ ideal of Haar null sets in a separable Banach space clearly contains the σ ideal of cube null sets, and the inclusion is easily seen to be proper if B is infinite dimensional. (For example, every compact subset of B is Haar null, while supports of cube measures are compact and not cube null. See, e.g., [2, Chapter 8].)

The next definition of null sets in Banach spaces comes from Aronszajn's attempt (see [1]) to give strong estimates of the size of points of Gâteaux non-differentiability of Lipschitz mappings. For any sequence $e_1, e_2, \ldots \in B$ he defines $\mathcal{A}(e_1, e_2, \ldots)$ as the family of those sets $E \subset B$ that can be written as a union of Borel sets E_n such that each E_n is null on every line in the direction e_n (i.e., for every $a \in B$, $\lambda\{t \in R : a + te_n \in E_n\} = 0$, where λ is the Lebesgue measure). The **Aronszajn null sets** are defined as those sets that belong to $\mathcal{A}(e_1, e_2, \ldots)$ whenever the span of the sequence $e_1, e_2, \ldots \in B$ is dense in B.

It can immediately be seen that the σ ideal of Aronszajn null sets is contained in the σ ideal of cube null sets. In addition, it has been observed by Phelps [6] that it is also contained in the σ ideal of **Gaussian null sets**; the latter being defined as the family of those Borel subsets of B that are null for every Gaussian measure. (See also [2, Chapter 8].)

It should be pointed out that the term 'Borel' in these definitions is essential. For example, if it is omitted from Aronszajn's definition, his σ ideal would no longer be proper even in case B is two dimensional. Indeed, a classical example of Sierpinski decomposes the plane (using the continuum hypothesis) into the union of two sets, one null on every horizontal line and one null on every vertical line! A similar paradoxical decomposition of infinitely dimensional spaces (using only axiom of choice) has been used by Bogachev [3] to show that the definitions of Aronszajn null sets and Gaussian null sets would trivially coincide if 'Borel' were replaced by 'universally Gaussian measurable'. However, it has been argued on several occasions (see, e.g., [7]) that the problem of coincidence of these classes should be answered for the original definitions. In this note we answer this problem by showing that:

THEOREM 1: *In every separable Banach space, the σ ideals of Aronszajn null sets, Gaussian null sets and cube null sets coincide.*

It was mentioned that the σ ideal of Aronszajn null sets is contained in the σ ideal of cube null sets and in the σ ideal of Gaussian null sets. We have to prove that for every Borel non-Aronszajn null set A there exist a non-degenerate cube measure μ and a non-degenerate Gaussian measure γ for which $\mu(A) > 0$ and $\gamma(A) > 0$. We prove a slightly more precise result:

THEOREM 2: *Let $e_1, e_2, \ldots \in B$ be a fixed sequence with dense span in B and $A \subset B$ a Borel set not belonging to $\mathcal{A}(e_1, e_2, \ldots)$. Then there exist a cube measure μ and a Gaussian measure γ for which $\mu(A) > 0$, $\gamma(A) > 0$, and μ and γ are the distributions of $a + \sum c_i X_i e_i$ and $b + \sum d_i Y_i e_i$, respectively, where c_i, d_i are positive real numbers, $\sum \|c_i e_i\| < \infty$, $\sum \|d_i e_i\| < \infty$, X_i are uniformly distributed random variables with values in $[0,1]$, Y_i are standard Gaussian random variables, and all the variables X_i, Y_i are independent.*

Definition 1: A system $e_1, e_2, \ldots \in B$, $e_1^*, e_2^*, \ldots \in B^*$ is called **orthogonal** if $\|e_i\| = 1$, $\langle e_i^*, e_j\rangle = \delta_{ij}$ for every i, j, and the span of the sequence $e_1, e_2, \ldots$ is dense in B.

LEMMA 1: *If the span of a sequence $e_1, e_2, \ldots \in B$ is dense in B, then there exists an orthogonal system $f_1, f_2, \ldots \in B$, $f_1^*, f_2^*, \ldots \in B^*$ for which $\bigcap_{i=1}^{\infty} \ker f_i^* = \{0\}$, and the linear spans of $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ are the same.*

Proof: Let $g_1^*, g_2^*, \ldots$ be non-zero elements of B^* for which $\sup_n \langle g_n^*, x\rangle = \|x\|$ for every $x \in B$. Then $\bigcap_{i=1}^{\infty} \ker g_i^* = \{0\}$.

We can define f_i, f_i^* by induction such that $\|f_i\| = 1$, $\langle f_i^*, f_j\rangle = \delta_{ij}$ for every i, j, for every i there exist j, k and l such that $e_i \in \operatorname{span}(f_1, f_2 \ldots f_j)$, $f_i \in \operatorname{span}(e_1, e_2 \ldots e_k)$ and $g_i^* \in \operatorname{span}(f_1^*, f_2^*, \ldots f_l^*)$. Then we have $\bigcap_{i=1}^{\infty} \ker f_i^* \subset \bigcap_{i=1}^{\infty} \ker g_i^* = \{0\}$. ∎

Definition 2: Let $e_1, e_2, \ldots \in B$, $e_1^*, e_2^*, \ldots \in B^*$ be an orthogonal system such that $\bigcap_{i=1}^{\infty} \ker e_i^* = \{0\}$, and let $I_n = [a_n, b_n] \subset \mathbb{R}$ be an enumeration of the rational intervals. For every sequence $\mathbf{s} \stackrel{\text{def}}{=} (s_1, s_2, \ldots, s_k)$ we define

$$C_{\mathbf{s}} \stackrel{\text{def}}{=} \{x \in B\colon \langle e_i^*, x\rangle \in I_{s_i}\ i = 1, 2, \ldots k\},$$

and for every Borel set E and for every $x \in B$ let

$$T_E^{\mathbf{s}}(x) \stackrel{\text{def}}{=} \{(r_1, r_2, \ldots, r_k) \in \mathbb{R}^k\colon x + r_1 e_1 + r_2 e_2 + \cdots + r_k e_k \in E \cap C_{\mathbf{s}}\}.$$

It is easy to see that the sets $T_E^{\mathbf{s}}(x)$ are Borel and hence we may also define a function $f_E^{\mathbf{s}}\colon B \to \mathbb{R}$ as follows:

$$f_E^{\mathbf{s}}(x) \stackrel{\text{def}}{=} \lambda(T_E^{\mathbf{s}}(x)).$$

LEMMA 2: *For every closed set E and for every $c \geq 0$ the set $\{x\colon f_E^{\mathbf{s}}(x) \geq c\}$ is closed.*

Proof: Suppose that $x_1, x_2, \dots$ is a sequence tending to x, and $f_E^{\mathbf{s}}(x_n) \geq c$ for every n. We have to show that $f_E^{\mathbf{s}}(x) \geq c$.

If for some $(r_1, r_2, \dots, r_k)$ we have $\langle e_i^*, x_n + r_1 e_1 + r_2 e_2 + \cdots + r_k e_k\rangle = \langle e_i^*, x_n\rangle + r_i \in I_{s_i}$, then r_i is in the interval $I_{s_i} \setminus \langle e_i^*, x_n\rangle$ of length at most $|I_{s_i}|$, and $x_n \to x$, thus $T_n \stackrel{\text{def}}{=} T_E^{\mathbf{s}}(x_n)$ is contained in a fixed cube of $\mathbb{R}^k$ if n is large enough. Since $\lambda(T_n) \geq c$ we have $\lambda(\limsup T_n) \geq c$. Now it is enough to show that for every $(r_1, r_2, \dots, r_k) \in \limsup T_n$ we have $x + r_1 e_1 + r_2 e_2 + \cdots + r_k e_k \in E \cap C_{\mathbf{s}}$, and this is immediate because there exists a subsequence $x_{n_1}, x_{n_2}, \dots$ for which $x_{n_m} + r_1 e_1 + r_2 e_2 + \cdots + r_k e_k \in E \cap C_{\mathbf{s}}$ and the set $E \cap C_{\mathbf{s}}$ is closed. ∎

COROLLARY 1: *For every Borel set E and for every sequence* s *the function $f_E^{\mathbf{s}}$ is Borel measurable.*

Proof: According to Lemma 2 this is clear for every closed set E. If $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of Borel sets, then the function $f^{\mathbf{s}}_{\bigcup_{n=1}^\infty E_n}$ is a limit of the functions $f^{\mathbf{s}}_{E_n}$. Similarly, if $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence then $f^{\mathbf{s}}_{\bigcap_{n=1}^\infty E_n}$ is a limit of the functions $f^{\mathbf{s}}_{E_n}$. Since every Borel set is obtained from closed sets by countably many increasing unions and decreasing intersections, and since limits of Borel measurable functions are Borel measurable, the function $f_E^{\mathbf{s}}$ is Borel measurable for every Borel set E. ∎

Definition 3: For every Borel set E, every $\mathbf{s} = (s_1, s_2, \dots, s_k)$ and for every non-negative real number c we define the set $E^{\mathbf{s}\,c}$ by induction, as follows.

If $s = \emptyset$, we let $E^{\mathbf{s}\,c} \stackrel{\text{def}}{=} E$. If for $\mathbf{s}^* = (s_1, s_2, \dots, s_{k-1})$ and c the set $E^{\mathbf{s}^*c}$ has been defined, then let

$$E^{\mathbf{s}\,c} \stackrel{\text{def}}{=} \{x \in E^{\mathbf{s}^*c}\colon f^{\mathbf{s}}_{E^{\mathbf{s}^*c}}(x) \geq c \prod_{i=1}^{k} |I_{s_i}|\}.$$

Remarks:

(1) If $E^{\mathbf{s}^*c}$ is closed, then applying Lemma 2 we infer that the set $E^{\mathbf{s}\,c}$ is also closed. Hence if E is closed then all the sets $E^{\mathbf{s}\,c}$ are closed as well.

(2) By a similar argument, from Corollary 1 we get that the sets $E^{\mathbf{s}\,c}$ are Borel for every Borel set E.

(3) If $E_1 \subset E_2$ and $c \geq d$, then $E_1^{\mathbf{s}\,c} \subset E_2^{\mathbf{s}\,d}$.

LEMMA 3: *Suppose that $1 \leq l < k$, $x_1, x_2 \in E^{s_1 s_2 \dots s_l c}$ and*

$$x_1 - x_2 \in \operatorname{span}(e_1, e_2, \dots, e_{l+1}).$$

Then $x_1 \in E^{\mathbf{s}\,c}$ *if and only if* $x_2 \in E^{\mathbf{s}\,c}$.

Proof: If $x_1 - x_2 \in \operatorname{span}(e_1, e_2, \dots, e_{m+1})$, say $x_1 = x_2 + t_1e_1 + t_2e_2 + \cdots + t_{m+1}e_{m+1}$, and $x_1, x_2 \in E^{s_1 s_2 \dots s_m c}$ for some $l \le m < k$, then

$$x_1 + r_1e_1 + r_2e_2 + \cdots + r_{m+1}e_{m+1} =$$
$$x_2 + (r_1+t_1)e_1 + (r_2+t_2)e_2 + \cdots + (r_{m+1}+t_{m+1})e_{m+1},$$

thus the sets $T^{s_1 s_2 \cdots s_{m+1}}_{E^{s_1 s_2 \cdots s_m c}}(x_1)$ and $T^{s_1 s_2 \cdots s_{m+1}}_{E^{s_1 s_2 \cdots s_m c}}(x_2)$ are translated copies of each other. Hence these sets are of the same measure with respect to λ, that is, $x_1 \in E^{s_1 s_2 \cdots s_{m+1} c}$ if and only if $x_2 \in E^{s_1 s_2 \cdots s_{m+1} c}$. By induction we have that $x_1 \in E^{\mathbf{s}\,c}$ if and only if $x_2 \in E^{\mathbf{s}\,c}$. ∎

COROLLARY 2: *If* $x \in E^{\mathbf{s}\,c}$ *then* $f^{\mathbf{s}}_{E^{\mathbf{s}\,c}}(x) \ge c\prod_{i=1}^{k} |I_{s_i}|$.

Proof: Applying Lemma 3 for $l = k-1$, we have

$$T^{\mathbf{s}}_{E^{\mathbf{s}\,c}}(x) = T^{\mathbf{s}}_{E^{\mathbf{s}^*}c}(x),$$

thus

$$f^{\mathbf{s}}_{E^{\mathbf{s}\,c}}(x) = f^{\mathbf{s}}_{E^{\mathbf{s}^*}c}(x) \ge c\prod_{i=1}^{k} |I_{s_i}|. \qquad ∎$$

LEMMA 4: *If for a sequence* $s_1, s_2, \dots$ *and* $c > 0$ *we have* $\sum |I_{s_i}| < \infty$ *and the intersection*

$$\bigcap_{k=1}^{\infty} E^{s_1 s_2 \cdots s_k c}$$

is non-empty and closed, then there exists a non-degenerate cube measure μ *for which* $\mu(E) > 0$.

Proof: Let

$$a \in \bigcap_{k=1}^{\infty} E^{\mathbf{s}\,c} \stackrel{\text{def}}{=} F.$$

Now, we have

$$\bigcap_k C_{\mathbf{s}} = \{a + \sum_k x_k e_k \colon \langle e_k^*, a\rangle + x_k \in I_{s_k}\}.$$

Indeed, one of the inclusions is trivial, and the other follows from $\bigcap \ker e_k^* = \{0\}$.

Let

$$S_k \stackrel{\text{def}}{=} \{a + y \in E^{\mathbf{s}\,c} \colon y \in \operatorname{span}(e_1, e_2, \dots, e_k)\}.$$

Applying Lemma 3, we have $S_l \subset \bigcap_k E^{\mathbf{s}\,c}$ for every l, and since F is closed,

$$F \supset \bigcap_k (S_k + \operatorname{span}(e_{k+1}, e_{k+2}, \ldots))$$

holds. Since $\lambda(S_k) \geq c \prod_{i=1}^{k} |I_{s_k}|$, we have $\mu(S_k + \operatorname{span}(e_{k+1}, e_{k+2}, \ldots)) \geq c$ for the cube measure μ determined by $\bigcap_k C_{\mathbf{s}}$, i.e. for the distribution of

$$(a - \sum_k \langle e_k^*, a\rangle) + \sum_i X_i |I_{s_i}| e_i$$

where X_i are independent random variables of uniform distributions with values in $[0,1]$. Thus we obtain $\mu(F) \geq c$. ∎

LEMMA 5: *If $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of Borel sets, $E \subset \bigcup_{n=1}^{\infty} E_n$, and $c > d$, then for any* $\mathbf{s}$

$$E^{\mathbf{s}\,c} \subset \bigcup_{n=1}^{\infty} E_n^{\mathbf{s}\,d}.$$

If $F_1 \supset F_2 \supset \cdots$ is a decreasing sequence of Borel sets and $F = \bigcap_{n=1}^{\infty} F_n$, then for any c and $\mathbf{s}$

$$F^{\mathbf{s}c} = \bigcap_{n=1}^{\infty} F_n^{\mathbf{s}\,c}.$$

Proof: We prove the statement by induction with respect to the length of the sequence $\mathbf{s}$. Clearly $E^{\emptyset\,c} \subset \bigcup_n E_n^{\emptyset\,d}$ and $F^{\emptyset\,c} = \bigcap_n F_n^{\emptyset\,c}$. Assume that $E^{\mathbf{s}^*c} \subset \bigcup_n E_n^{\mathbf{s}^*\,d}$ and $F^{\mathbf{s}^*c} = \bigcap_n F_n^{\mathbf{s}^*c}$ for a given $\mathbf{s}^* \stackrel{\text{def}}{=} (s_1, s_2, \ldots, s_{k-1})$. Then for $\mathbf{s} = (s_1, s_2, \ldots, s_k)$, we have

$$T^{\mathbf{s}}_{E^{\mathbf{s}^*c}}(x) \subset \bigcup_{n=1}^{\infty} T^{\mathbf{s}}_{E_n^{\mathbf{s}^*d}}(x) \quad \text{and} \quad T^{\mathbf{s}}_{F^{\mathbf{s}^*c}}(x) = \bigcap_{n=1}^{\infty} T^{\mathbf{s}}_{F_n^{\mathbf{s}^*c}}(x)$$

for every $x \in B$, thus

$$f^{\mathbf{s}}_{E^{\mathbf{s}^*c}} \leq \lim_{n\to\infty} f^{\mathbf{s}}_{E_n^{\mathbf{s}^*d}} \quad \text{and} \quad f^{\mathbf{s}}_{F^{\mathbf{s}^*c}} = \lim_{n\to\infty} f^{\mathbf{s}}_{F_n^{\mathbf{s}^*c}},$$

where the sequence $f^{\mathbf{s}}_{E_n^{\mathbf{s}^*d}}$ is increasing and the sequence $f^{\mathbf{s}}_{F_n^{\mathbf{s}^*c}}$ is decreasing. By this we have

$$E^{\mathbf{s}\,c} \subset \bigcup_{n=1}^{\infty} E_n^{\mathbf{s}d} \quad \text{and} \quad F^{\mathbf{s}c} = \bigcap_{n=1}^{\infty} F_n^{\mathbf{s}\,c}. \qquad ∎$$

LEMMA 6: *For every sequence* $\mathbf{s} = (s_1, s_2, \dots, s_k)$, *for every positive real numbers* ε *and* $1 > c > d$, *and for every Borel set* E *we have*

$$E^{\mathbf{s}\,c} \setminus \Big(\bigcup_{\sigma:\, |I_\sigma| < \varepsilon} E^{(\mathbf{s},\sigma)d} \Big) \in \mathcal{A}(e_{k+1}).$$

Proof: Let C be defined as the set of the points $x \in E^{\mathbf{s}\,c}$ for which $\langle e^*_{k+1}, x\rangle$ is not a Lebesgue density point of the e^*_{k+1} image of the set

$$E^{\mathbf{s}\,c} \cap (x + \operatorname{span}(e_1, e_2, \cdots, e_{k+1})),$$

i.e. not a density point of the set

$$M_x = \{r\colon \exists y_r = x + r_1 e_1 + r_2 e_2 + \cdots + r_{k+1} e_{k+1} \in E^{\mathbf{s}\,c},\ \langle e^*_{k+1}, y_r\rangle = r\}.$$

Applying the Lebesgue density theorem, we immediately see that C is null on every line in the direction e_{k+1}, therefore we have to show that for every $x \in E^{\mathbf{s}\,c} \setminus C$ there exists a σ for which $x \in E^{(\mathbf{s},\sigma)d}$. Indeed, since Remark 2 shows that the set $E^{\mathbf{s}\,c} \setminus (\bigcup_{\sigma:\, |I_\sigma|<\varepsilon} E^{(\mathbf{s},\sigma)d})$ is Borel, it would then belong to $\mathcal{A}(e_{k+1})$.

Since $\langle e^*_{k+1}, x\rangle$ is a density point of M_x and $\frac{d}{c} < 1$, it can be covered by an arbitrarily small rational interval I_σ such that the measure of the points of the interval that belong to our set is at least $\frac{d}{c}|I_\sigma|$, that is, there exists a σ such that $\langle e^*_{k+1}, x\rangle \in I_\sigma$, $|I_\sigma| < \varepsilon$, and the measure of the set

$$\{r \in I_\sigma\colon \exists y_r = x + r_1 e_1 + r_2 e_2 + \cdots + r_{k+1} e_{k+1} \in E^{\mathbf{s}\,c},\ \langle e^*_{k+1}, y_r\rangle = r\}$$

is at least $\frac{d}{c}|I_\sigma|$. We fix $s_{k+1} = \sigma$. Now, for every fixed r from the above set (i.e. for every fixed r_{k+1}) we choose the corresponding $y_r \in E^{\mathbf{s}\,c}$, and applying Corollary 2 we have

$$f^{\mathbf{s}}_{E^{\mathbf{s}\,c}}(y_r) \geq c \prod_{i=1}^{k} |I_{s_i}|.$$

This means that the measure of the set

$$\{(t_1, t_2, \dots, t_k) \in \mathbb{R}^k\colon y_r + t_1 e_1 + t_2 e_2 + \cdots + t_k e_k \in E^{\mathbf{s}\,c} \cap C_{\mathbf{s}}\}$$

is at least $c \prod_{i=1}^{k} |I_{s_i}|$. Thus the measure of the translated copy

$$\{(t^*_1, t^*_2, \dots, t^*_k) \in \mathbb{R}^k\colon x + t^*_1 e_1 + t^*_2 e_2 + \cdots + t^*_k e_k + r_{k+1} e_{k+1} \in E^{\mathbf{s}\,c} \cap C_{\mathbf{s}}\}$$

is at least $c \prod_{i=1}^{k} |I_{s_i}|$, and the Fubini theorem gives that the measure of the set

$$\{(t^*_1, t^*_2, \dots, t^*_k, r_{k+1}) \in \mathbb{R}^{k+1}\colon x + t^*_1 e_1 + t^*_2 e_2 + \cdots + t^*_k e_k + r_{k+1} e_{k+1} \in E^{\mathbf{s}\,c} \cap C_{(\mathbf{s},\sigma)}\}$$

is at least $c\prod_{i=1}^{k}|I_{s_i}|\cdot\frac{d}{c}|I_{s_{k+1}}|=d\prod_{i=1}^{k+1}|I_{s_i}|$. This set is a subset of

$$\{(t_1^*,t_2^*,\dots,t_k^*,r_{k+1})\in\mathbb{R}^{k+1}\colon x+t_1^*e_1+t_2^*e_2+\dots+t_k^*e_k+r_{k+1}e_{k+1}\in E^{\mathbf{s}\,d}\cap C_{(\mathbf{s},\sigma)}\},$$

hence we have

$$f_{E^{\mathbf{s}\,d}}^{(\mathbf{s},\sigma)}(x)\geq d\prod_{i=1}^{k+1}|I_{s_i}|,$$

that is, $x\in E^{(\mathbf{s},\sigma)d}$, as required. ∎

Lemma 7: *For every Borel set $A\notin\mathcal{A}(e_1,e_2,\dots)$ and for every $0<b<1$ there exist a closed subset $F\subset A$ and a sequence of integers $s_1,s_2,\dots$, such that $|I_{s_i}|<1/2^i$ and the intersection $\bigcap_k F^{s_1s_2\cdots s_kb}$ is non-empty and closed.*

Proof: It is enough to show that there exists a closed subset $F\subset A$ for which $\bigcap_k F^{\mathbf{s}c}\neq\emptyset$; because according to Remark 1 all the sets $F^{\mathbf{s}\,c}$ are closed, the set $\bigcap_k F^{\mathbf{s}c}$ is automatically closed.

It is a standard fact that for every Borel set A there exists a decomposition $A=\bigcup_{\mathbf{n}}\bigcap_k F_{n_1n_2\cdots n_k}$ such that

- the sets $F_{n_1n_2\cdots n_k}$ are closed;
- $F_{n_1n_2\cdots n_k}\supset F_{n_1n_2\cdots n_kn_{k+1}}$;
- $F_{n_1n_2\cdots n_k}\subset F_{n_1n_2\cdots n_{k-1}(n_k+1)}$;
- the sets $A_{n_1n_2\cdots n_k}\overset{\text{def}}{=}\bigcup_{\mathbf{m}}\bigcap_l F_{n_1n_2\cdots n_km_1m_2\cdots m_l}$ are Borel.

Indeed, writing F as a one-to-one continuous image of a closed set Z of sequences of integers, it suffices to define $F_{n_1n_2\cdots n_k}$ as the closure of the image of $Z_{n_1n_2\cdots n_k}$ (which is the set of those sequences from Z whose i-th term is at most n_i for $i\leq k$), and to observe that $A_{n_1n_2\cdots n_k}$ is the (one-to-one continuous) image of $Z_{n_1n_2\cdots n_k}$, so it is Borel. (A direct proof observing that the statement is trivial for closed sets and that it remains true after countable unions and intersections is also possible.)

Since applying Lemma 6 we have $E^{\mathbf{s}\,c}\setminus\bigcup_{\sigma\colon|I_\sigma|<\varepsilon}E^{(\mathbf{s},\sigma)d}\in\mathcal{A}(e_{k+1})$ for every Borel set E, $\varepsilon>0$ and $0<d<c<1$, there exists an $a\in A$ such that for every rational $\varepsilon>0$ and $0<d<c<1$, and for every possible $s_1,s_2,\dots,s_k$ and $n_1,n_2,\dots,n_k$,

$$a\notin A_{n_1n_2\cdots n_k}^{\mathbf{s}\,c}\setminus\bigcup_{\sigma:|I_\sigma|<\varepsilon}A_{n_1n_2\cdots n_k}^{(\mathbf{s},\sigma)d}$$

holds. We choose a sequence of rational numbers $1>r_0>r_1>r_2>\dots>c$. Now,

$$a\in A=\bigcup_{n_1}A_{n_1}\Longrightarrow\exists n_1\colon a\in A_{n_1}=A_{n_1}^{\emptyset\,r_0},$$

$$a \notin A_{n_1}^{\emptyset, r_0} \setminus \bigcup_{s_1 : |I_{s_1}| < 1/2} A_{n_1}^{s_1\, r_1} \Longrightarrow \exists s_1 \colon |I_{s_1}| < 1/2 \text{ and } a \in A_{n_1}^{s_1\, r_1},$$

$$A_{n_1 n_2} \nearrow A_{n_1} \overset{\text{Lemma 5}}{\Longrightarrow} A_{n_1}^{s_1\, r_1} \subset \bigcup_{n_2} A_{n_1 n_2}^{s_1\, r_2} \Longrightarrow \exists n_2 \colon a \in A_{n_1 n_2}^{s_1\, r_2},$$

$$a \notin A_{n_1 n_2}^{s_1\, r_2} \setminus \bigcup_{s_2 : |I_{s_2}| < 1/4} A_{n_1 n_2}^{s_1 s_2\, r_3} \Longrightarrow \exists s_2 \colon |I_{s_2}| < 1/4 \text{ and } a \in A_{n_1 n_2}^{s_1 s_2\, r_3},$$

$$A_{n_1 n_2 n_3} \nearrow A_{n_1 n_2} \overset{\text{Lemma 5}}{\Longrightarrow} A_{n_1 n_2}^{s_1 s_2\, r_3} \subset \bigcup_{n_3} A_{n_1 n_2 n_3}^{s_1 s_2\, r_4} \Longrightarrow \exists n_3 \colon a \in A_{n_1 n_2 n_3}^{s_1 s_2\, r_4}.$$

Following this procedure (and applying Remark 3) we have

$$a \in \bigcap_k A_{n_1 n_2 \cdots n_k}^{s_1 s_2 \cdots s_k\, c},$$

and

$$\bigcap_k A_{n_1 n_2 \cdots n_k}^{s_1 s_2 \cdots s_k\, c} \subset \bigcap_k F_{n_1 n_2 \cdots n_k}^{s_1 s_2 \cdots s_k\, c} \subset F$$

where

$$F \overset{\text{def}}{=} \bigcap_k F_{n_1 n_2 \cdots n_k}.$$

Now it is immediate that F is a non-empty closed subset of A, and, in view of Lemma 5, property $F_{n_1} \supset F_{n_1 n_2} \supset \cdots$ implies

$$\bigcap_k F^{\mathbf{s}\, c} = \bigcap_k \Big(\bigcap_l F_{n_1 n_2 \ldots n_l}\Big)^{\mathbf{s}\, c} = \bigcap_k \bigcap_l F_{n_1 n_2 \ldots n_l}^{\mathbf{s}\, c} \ni a. \qquad \blacksquare$$

Lemma 8: *Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers for which $\sum_n \varepsilon_n < \infty$ and $0 < \varepsilon_n < 1$ for every n. If $A \subset \prod_{n=1}^{\infty} [\varepsilon_n, 1]$ has positive Lebesgue measure in $[0,1]^{\mathbb{N}}$ and μ is a measure in $\mathbb{R}^{\mathbb{N}}$ for which $\mu(\prod_{n=1}^{\infty}[0, \varepsilon_n]) > 0$, then there exists a $\mathbf{t} \in [0,1]^{\mathbb{N}}$ such that for the measure $\mu_{\mathbf{t}}(X) \overset{\text{def}}{=} \mu(X + \mathbf{t})$ we have $\mu_{\mathbf{t}}(A) > 0$.*

Proof:

$$\begin{aligned}
\int_{[0,1]^{\mathbb{N}}} \mu_{\mathbf{t}}(A)\, d\lambda(\mathbf{t}) &\geq \int_{[0,1]^{\mathbb{N}}} \int_{[0,1]^{\mathbb{N}}} \chi_{A-\mathbf{t}}(\mathbf{u})\, d\mu(\mathbf{u})\, d\lambda(\mathbf{t}) \\
&= \int_{[0,1]^{\mathbb{N}}} \int_{[0,1]^{\mathbb{N}}} \chi_{A-\mathbf{t}}(\mathbf{u})\, d\lambda(\mathbf{t})\, d\mu(\mathbf{u}) \\
&= \int_{[0,1]^{\mathbb{N}}} \int_{[0,1]^{\mathbb{N}}} \chi_{A-\mathbf{u}}(\mathbf{t})\, d\lambda(\mathbf{t})\, d\mu(\mathbf{u}) \\
&= \int_{[0,1]^{\mathbb{N}}} \lambda((A - \mathbf{u}) \cap [0,1]^{\mathbb{N}})\, d\mu(\mathbf{u}) \\
&\geq \int_{\prod [0, \varepsilon_n]} \lambda((A - \mathbf{u}) \cap [0,1]^{\mathbb{N}})\, d\mu(\mathbf{u}) \\
&= \mu\Big(\prod [0, \varepsilon_n]\Big) \lambda(A) > 0,
\end{aligned}$$

thus there exists a $\mathbf{t}$ for which $\mu_{\mathbf{t}}(A) > 0$. ∎

Now we are ready to prove Theorem 2.

Proof of Theorem 2: If $\{e_i\}$ and $\{f_j\}$ are two sequences such that each e_i is in the linear span of finitely many f_j's, then $\mathcal{A}(e_1, e_2, \ldots) \subset \mathcal{A}(f_1, f_2, \ldots)$. Hence, if a Borel set A does not belong to $\mathcal{A}(f_1, f_2, \ldots)$ then applying Lemma 1 there exists an orthogonal system $e_1, e_2, \ldots \in B$, $e_1^*, e_2^*, \ldots \in B^*$ such that the linear spans of $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ are the same and $A \notin \mathcal{A}(e_1, e_2 \ldots)$.

Now, applying Lemma 4 and Lemma 7 we have a non-degenerate cube measure μ for which $\mu(A) > 0$ and it can be seen from the proofs that μ is the distribution of a random variable of the form $a + \sum c_i X_i e_i$. Since each f_i is in the linear span of finitely many e_j's, for numbers c_i^* small enough the support of the distribution of $(a + \sum \frac{1}{2} c_i e_i) + \sum c_i^* X_i f_i$ is contained in the support of the distribution of $a + \sum c_i X_i e_i$, moreover, applying Lemma 8 we have $\mu^*(A) > 0$ for the cube measure μ^* which is the distribution of $a^* + \sum c_i^* X_i f_i$ with a suitable point a^* and small numbers c_i^*.

Finally, applying Lemma 8 again we can choose very small numbers d_i and point b such that $\gamma(A) > 0$ for the distribution of $b + \sum d_i Y_i f_i$ (where Y_i are independent standard Gaussian variables). ∎

Remark: In the spirit of [1] and [5] it would be natural to state and prove our results in Fréchet spaces; this would require only obvious minor changes in the arguments, since for our purposes cube and Gaussian measures may be defined, e.g., as distributions of a.s. convergent sums $a + \sum X_i e_i$ (where X_i are independent $[0, 1]$ valued uniformly distributed random variables) and $a + \sum Y_i e_i$ (where Y_i are independent standard Gaussian variables), respectively.

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